

Asymptotic completeness for the massless spin-boson model

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Abstract

We consider generalized versions of the massless spin-boson model. Building on the recent work in [DRK12] and [DRK14], we prove asymptotic completeness.

1 Introduction

This paper is concerned with the scattering theory for generalized spin-boson models with massless bosons. That is, we consider a spin system (an “atom”) coupled to a scalar field of quantized massless bosons. With the help of previously established properties, such as relaxation to the ground state and a uniform bound on the number of soft bosons, we now show that asymptotic completeness holds provided the excited states of the uncoupled system of spin and bosons have finite life times once the interaction is turned on. (Fermi-Golden Rule condition.)

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To describe our main result and its proof we now introduce the system in some detail. We confine ourselves to a concrete system satisfying all our assumptions. More general hypotheses are described in the next section. Our model consists of a small system (atom, spin) coupled to a free bosonic field. The Hilbert space of the total system is

$$\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_F$$

where $\mathcal{H}_S = \mathbb{C}^n$ for some $n < \infty$ (S for “small system”) and the field space \mathcal{H}_F is the symmetric Fock space over $L^2(\mathbb{R}^3)$. The total Hamiltonian is of the form

$$H = H_S \otimes \mathbb{1} + \mathbb{1} \otimes H_F + H_I$$

where H_S is a hermitian matrix with simple eigenvalues only, and H_F denotes the Hamiltonian of the free *massless* field. The coupling operator H_I is of the form

$$H_I = \lambda D \otimes \int_{\mathbb{R}^3} (\hat{\phi}(k) a_k^* + \overline{\hat{\phi}(k)} a_k) dk.$$

Here $D = D^*$ is a matrix acting on \mathcal{H}_S , $\lambda \in \mathbb{R}$ is a sufficiently small coupling constant, and a_k^*, a_k are the usual creation and annihilation operators of a mode $k \in \mathbb{R}^3$ satisfying the “Canonical Commutation Relations”. The function $\hat{\phi} \in L^2(\mathbb{R}^3)$ is a “form factor” that imposes some infrared regularity and an ultraviolet cutoff: simple and sufficient assumptions are that $\hat{\phi}$ has compact support and that $\hat{\phi} \in C^3(\mathbb{R}^3 \setminus \{0\})$ with

$$\hat{\phi}(k) = |k|^{(\alpha-1)/2}, \quad |k| \leq 1,$$

for some $\alpha > 0$. This assumption ensures, e.g., that H has a unique ground state Ψ_{gs} . To rule out the existence of excited bound states we assume the Fermi-Golden Rule condition stated in the next section. Further spectral input is not needed but our results do certainly have non-trivial consequences for the spectrum of H .

Under the time-evolution generated by H it is expected that every excited state relaxes to the ground state by emission of photons whose dynamics is asymptotically free. The existence of excited states with this property is well known [FGS01, GZ09]; they are spanned by products of asymptotic creation operators applied to the ground state Ψ_{gs} , that is, by vectors of the form:

$$a_+^*(f_1) \dots a_+^*(f_m) \Psi_{\text{gs}} = \lim_{t \rightarrow \infty} e^{i(H-E)t} a^*(f_{1,t}) \dots a^*(f_{m,t}) \Psi_{\text{gs}},$$

where $f_t = e^{-i\omega t} f$ and $\omega(k) = |k|$. Asymptotic completeness of Rayleigh scattering means that the span of these vectors is dense in \mathcal{H} . In particular, the representation of the CCR given by the asymptotic creation and annihilation is equivalent to the Fock representation; non-Fock representation such as those discussed in [DG04] do not occur.

Asymptotic completeness of Rayleigh scattering is expected to hold for a large class of models of atoms interacting with quantized radiation. Yet, despite considerable efforts [HS95, DG99, FGS02a, Gér02], it has so far resisted a rigorous proof. The main stumbling blocks were the lack of time-independent photon bounds and a the lack of a quantitative understanding of a property called *relaxation to the ground state* (the zero temperature analog of return to equilibrium), which is known to follow from asymptotic completeness [FGS01]. For the spin-boson model, these problems were solved in the previous papers [DRK12, DRK14]. We now use these results and expand on them to give a complete proof of asymptotic completeness for a fairly large class of *massless* spin-bosons system. Independently from us Faupin and Sigal had embarked on a similar project using results from [DRK12]. They obtain AC for a class of spin-boson models very similar to ours.

Our proof of asymptotic completeness relies on methods and tools developed for the purpose of establishing the relaxation to equilibrium (and non-equilibrium steady states) of systems at positive temperature. These methods are based on analogues with one-dimensional statistical mechanics. In fact, by a cluster expansion we have previously shown (in [DRK12]) that a weak form of relaxation to the ground state holds, by which we mean that

$$\lim_{t \rightarrow \infty} \langle \Psi_t, O \Psi_t \rangle = \langle \Psi_{\text{gs}}, O \Psi_{\text{gs}} \rangle \quad (1.1)$$

for a suitable C^* algebra of observables O . Our second key tool, established by the same cluster expansion, states that the number of soft photons emitted in the process of relaxation is uniformly bound in time. More precisely, there exists a constant $\kappa > 0$ such that

$$\sup_t \langle \Psi_t, e^{\kappa N} \Psi_t \rangle < \infty \quad (1.2)$$

where N denotes the number operator on Fock space. Our third key ingredient concerns the number of bosons in a ball $|x| < vt$ where $v < 1$, the speed of light. We show that

$$\langle \Psi_t, d\Gamma(\theta_t) \Psi_t \rangle = \langle \Psi_{\text{gs}}, d\Gamma(\theta_t) \Psi_{\text{gs}} \rangle + \mathcal{O}(\langle t \rangle^{-\alpha}) \quad (1.3)$$

where θ_t denotes a smoothed characteristic function of the set $\{x \in \mathbb{R}^3 \mid |x| \leq vt\}$ with some $v < 1$. Moreover, if θ_t is replaced by θ_{t_c} then (1.3) holds uniformly in t_c with $|\lambda|^{-2} \leq t_c \leq t$. This propagation bound and soft-photon bound were established in [DRK14] by a slight variation of the cluster expansion mentioned above (in the case of the soft-photon bound, this adjustment was even unnecessary and one could have simply copied the treatment of [DRK12], as was also remarked in [FS12a])

Our basic strategy for proving asymptotic completeness is the usual one from time-dependent scattering theory [SS87, Gra90]. In the present context this means

that we construct a suitable (right-)inverse Z of the wave operator W_+ define by

$$W_+ a^*(f_1) \dots a^*(f_m) \Omega = a_+^*(f_1) \dots a_+^*(f_m) \Psi_{gs}.$$

The identity $W_+ Z = \mathbb{1}$ in \mathcal{H} shows that W_+ has full range, which is equivalent to asymptotic completeness as explained above. Our tools, however, are certainly not the usual ones: the properties (1.1)-(1.3) augmented by a strong form of the local relaxation (1.1), see Proposition 4.1, are enough for establishing both the existence of Z and the identity $W_+ Z = \mathbb{1}$. Important standard tools from scattering theory, such as propagation estimates and Mourre estimates, are not needed in the present work.

We conclude this introduction with a discussion of previous work on Rayleigh scattering and asymptotic completeness (AC) in models similar to ours. In '97 Spohn considered an electron that is bound by a perturbed harmonic potential and coupled to the quantized radiation field in dipole approximation [Spo97]. Knowing AC for the harmonically bound particle, a result due to Arai [Ara83], he concludes AC in his perturbed model by summing a Dyson series for the inverse wave operator. This is the first proof of AC in a non-solvable model with massless bosons. AC in more general models of a bound particle coupled to massive bosons was established by [DG99] and by [FGS02a]. These works adapt the methods from many-body quantum scattering, i.e. Mourre estimates and propagation estimates, to non-relativistic QFT. Consequently the Mourre estimate for massless bosons was a main concern of subsequent work [Ski98, GGM04, FGS08]. For interesting partial results on massless boson scattering assuming a Mourre estimate we refer to [Gér02].

Most recently, in [FS12b, FS12a], Faupin and Sigal succeeded in establishing AC for a large class of models of atoms interacting with quantized radiation. They assume a time-independent bound on $\langle \Psi_t, d\Gamma(1/|k|)^2 \Psi_t \rangle$. Based on this assumption they establish AC for a large variety of models including non-relativistic QED and spin-boson models. Following the strategy in [DRK12], a bound on $\langle \Psi_t, d\Gamma(1/|k|)^2 \Psi_t \rangle$ can be obtained in the same way as a bound on $\langle \Psi_t, N^2 \Psi_t \rangle$ upon slightly strengthening the infrared assumption, as is remarked in [FS12a]. For a certain class of spin-boson models (similar to ours), Faupin and Sigal have thus proven AC prior to us. However, given the methods and results from [DRK12], which are needed by them as well, we believe our approach to AC for spin-boson models is simpler.

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2 Assumptions and Results

We now describe the class of spin-boson models considered in this paper along with all assumptions. It is instructive to do this in d rather than 3 space dimension, although we shall later confine ourselves to 3 dimensions for simplicity. This section also serves us for collecting frequently used notations.

2.1 Notations

The symmetric Fock-space over a one-particle space \mathfrak{h} is denoted by $\Gamma(\mathfrak{h})$ which is defined by

$$\Gamma(\mathfrak{h}) = \bigoplus_{n=0}^{\infty} P_S \mathfrak{h}^{\otimes n} \quad (2.1)$$

with P_S is the projection to symmetric tensors and $\mathfrak{h}^{\otimes 0} \equiv \mathbb{C}$. Hence $\mathcal{H}_F = \Gamma(\mathfrak{h})$ with $\mathfrak{h} = L^2(\mathbb{R}^d)$. We define the finite-photon space

$$\mathcal{D}_{fin} := \bigcup_{n \in \mathbb{N}} \mathbb{1}_{N \leq n} \mathcal{H}_F, \quad (2.2)$$

and we write also \mathcal{D}_{fin} for $\mathcal{H}_S \otimes \mathcal{D}_{fin}$.

By $a^*(f)$ and $a(f)$ we denote the smeared creation and annihilation operators in \mathcal{H}_F that are related to a_k^* and a_k by

$$a^*(f) = \int f(k) a_k^* dk, \quad a(f) = \int \overline{f(k)} a_k dk.$$

Their sum is the self-adjoint field operator $\Phi(f) = a^*(f) + a(f)$, which generates the Weyl-operator

$$\mathcal{W}(f) = e^{i\Phi(f)}. \quad (2.3)$$

Our assumptions on form factor and initial states are conveniently expressed in terms of the following spaces:

Definition 2.1. For $0 < \alpha < 1$ the subspace $\mathfrak{h}_\alpha \subset \mathfrak{h}$ consists of all $\psi \in \mathfrak{h}$ such that $\hat{\psi} \in C^3(\mathbb{R}^d \setminus \{0\})$ has compact support, and, for all multi-indices m with $|m| \leq 3$,

$$|\partial_k^m \hat{\psi}(k)| \leq C |k|^{(\beta-d+2)/2-|m|} \quad (2.4)$$

for some $\beta > \alpha$ and $C < \infty$. By an easy application of Lemma A.1 of [DRK14], any $\psi \in \mathfrak{h}_\alpha$ satisfies

$$|\psi(x)| \leq C(1 + |x|)^{-\min\{\frac{d+\beta+2}{2}, 3\}} \quad (2.5)$$

for some $\beta > \alpha$.

The subspace $\mathcal{D}_\alpha \subset \mathcal{H}$ is defined by

$$\mathcal{D}_\alpha := \text{Span}\{\psi_S \otimes \mathcal{W}(f)\Omega \mid \psi_S \in \mathcal{H}_S, f \in \mathfrak{h}_\alpha\}. \quad (2.6)$$

It is dense in \mathcal{H} because \mathfrak{h}_α is dense in \mathfrak{h} .

The following notation is very convenient and often used in this paper: If $\Psi \in \mathcal{H} = \mathcal{H}_a \otimes \mathcal{H}_b$ and $\gamma \in \mathcal{H}_a$, then $\Psi_\gamma = \langle \gamma | \Psi \in \mathcal{H}_b$ is defined by

$$\eta \otimes \langle \gamma | \Psi = (|\eta\rangle \langle \gamma| \otimes \mathbb{1})\Psi, \quad \text{for } \eta \in \mathcal{H}_a.$$

It follows that $\langle \eta \otimes \Psi_\gamma, \Phi \rangle = \langle \Psi_\gamma, \Phi_\eta \rangle$ and $\|\langle \gamma | \Psi\|^2 = \langle \Psi, (|\gamma\rangle \langle \gamma| \otimes \mathbb{1})\Psi \rangle$.

2.2 Model and Assumptions

Recall from the introduction that the system we consider is described by a Hamiltonian H on $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_F$ that is of the form

$$H = H_S \otimes \mathbb{1} + \mathbb{1} \otimes H_F + H_I. \quad (2.7)$$

The free field operator H_F and the interaction operator H_I are given by

$$\begin{aligned} H_F &= d\Gamma(\omega) = \int_{\mathbb{R}^d} \omega(k) a_k^* a_k dk, \quad \omega(k) = |k|, \\ H_I &= \lambda D \otimes \Phi(\phi). \end{aligned}$$

In the following we describe our assumptions on the matrices H_S , D and the form factor ϕ . These choices and assumptions are assumed to hold throughout the article with the exception of the present Section 2, where we will state some earlier results that do not require these strong assumptions. The infrared (small Fourier mode k) behavior of the form factor determines temporal correlations in the model and some regularity near $k = 0$ is needed:

Assumption 2.1 (α -Infrared regularity). The form factor ϕ is in \mathfrak{h}_α and the dimension $d = 3$.

Of course, the restriction to $d = 3$ is not really necessary provided one modifies slightly the infrared assumption but we prefer to keep it for simplicity. Our second assumption ensures that the coupling is effective:

Assumption 2.2 (Fermi Golden Rule). We assume that the spectrum of H_S is non-degenerate (all eigenvalues are simple) and we let $e_0 := \min \sigma(H_S)$ (atomic

ground state energy). Most importantly, we assume that for any eigenvalue $e \in \sigma(H_S)$, $e \neq e_0$, there is a set $\{e_i\}_{i=1}^n$ of eigenvalues such that

$$e = e_n > e_{n-1} > \dots > e_1 > e_0, \quad (2.8)$$

and for all $i = 0, \dots, n-1$,

$$\text{Tr}[P_i D P_{i+1} D P_i] \int_{\mathbb{R}^d} dk \delta(|k| - e_i - e_{i+1}) |\hat{\phi}(k)|^2 > 0. \quad (2.9)$$

Here P_i denotes the projection onto the eigenspace associated with e_i . The right hand side is well-defined since $\hat{\phi}$ is continuous.

2.3 Results

The following results on the ground state and its properties are prerequisites for our definition of the wave operator.

Theorem 2.3 ([BFS98, Gér00]). *Let the form factor ϕ satisfy $\phi/\omega \in \mathfrak{h}$. Then the ground-state energy*

$$E_{\text{gs}} = \inf_{\|\Psi\|=1} \langle \Psi, H \Psi \rangle \quad (2.10)$$

is finite; $E_{\text{gs}} \leq \inf H_S$, and there is a vector $\Psi_{\text{gs}} \in \mathcal{H}$ such that

$$H \Psi_{\text{gs}} = E_{\text{gs}} \Psi_{\text{gs}} \quad (2.11)$$

If the coupling constant $|\lambda|$ is sufficiently small, then such a Ψ_{gs} is unique (up to scalar multiplication).

The uniqueness of the ground state for small $|\lambda|$ follows from the simplicity of the eigenvalue $e_0 := \min \sigma(H_S)$ by an overlap argument that is probably due to [BFS98].

Additionally, we need an exponential localization property of photons, namely:

Lemma 2.4. *Assume Assumptions 2.1 and 2.2 and let $|\lambda|$ be sufficiently small. Then, for some $\kappa > 0$,*

$$\langle \Psi_{\text{gs}}, e^{\kappa N} \Psi_{\text{gs}} \rangle < \infty. \quad (2.12)$$

This property is proven in this paper, at the end of Section 4. Earlier results, exhibiting somewhat weaker localization, can be found in [Hir03].

To describe scattering, let us introduce the standard identification operator $I : \mathcal{H}_{\text{F}} \rightarrow \mathcal{H}$

$$I a^*(f_1) a^*(f_2) \dots a^*(f_m) \Omega := a^*(f_1) a^*(f_2) \dots a^*(f_m) \Psi_{\text{gs}} \quad (2.13)$$

for f_1, \dots, f_m such that $(1 + |k|^{-1})f_j \in \mathfrak{h}$, and then extended to a closed operator, see e.g. [HS95]. Note that the adjoint of I is densely defined and hence I is closable. The well-known Cook's argument yields the following standard result:

Theorem 2.5. *The wave operators*

$$W_{\pm}\Psi := \lim_{t \rightarrow \infty} W_t\Psi, \quad W_t\Psi = e^{\mp itH} I e^{\pm it(E_{\text{gs}} + H_{\text{F}})} \Psi \quad (2.14)$$

exist for Ψ in a dense subset of \mathcal{D}_{fin} . They satisfy $\|W_{\pm}\Psi\| = \|\Psi\|$ and therefore W_{\pm} extends to an isometry $\mathcal{H}_{\text{F}} \rightarrow \mathcal{H}$.

Existence of wave operators for a large class of models with massive and massless photons is established in [HK69, DG99, FGS01, GZ09].

Our main result complements the former statement by establishing

Theorem 2.6 (Asymptotic completeness). *Assume Assumptions 2.1 and 2.2. Then there is a $\lambda_0 > 0$ such that for any coupling strength λ satisfying $0 < |\lambda| \leq \lambda_0$;*

$$\text{Ran} W_{\pm} = \mathcal{H}. \quad (2.15)$$

Remark 2.7. *We will only discuss W_+ . To treat W_- , it suffices to define a time-reversal operator (antilinear involution)*

$$\Theta = \Theta_{\text{S}} \otimes \Theta_{\text{F}} \quad (2.16)$$

where Θ_{S} is complex conjugation in a basis diagonalizing H_{S} , and $\Theta_{\text{F}} = \Gamma(\theta_{\text{F}})$ with $\theta_{\text{F}}\psi(q) = \psi(-q)$. Then we check that, if H satisfies our assumptions, then $\Theta H \Theta$ satisfies them as well.

3 Technical tools

In this section, we list all the tools that we use which are not proven in the present paper, or only in the appendix. By smooth indicators θ , we mean spherically symmetric functions in $C_0^\infty(\mathbb{R}^d)$, i.e. $\theta(x) = \theta(|x|)$, with $0 \leq \theta \leq 1$, and we write $\theta_t(x) = \theta(x/t)$. All the following lemmata hold for sufficiently small λ , uniformly in λ . From now on, we always assume that Assumptions 2.1 and 2.2 hold. Moreover, we write throughout this section

$$\Psi_t = e^{-itH} \Psi_0 \quad \text{with} \quad \Psi_0 \in \mathcal{D}_{\alpha}.$$

The first tool is the weak relaxation to the ground state.

Lemma 3.1 (Weak relaxation [DRK12]). *Let the C^* -algebra \mathcal{W}_{α} be generated by $S \in \mathcal{B}(\mathcal{H}_{\text{S}})$ and $\mathcal{W}(\psi)$, $\psi \in \mathfrak{h}_{\alpha}$. For any $O \in \mathcal{W}_{\alpha}$, we have*

$$\lim_{t \rightarrow \infty} \langle \Psi_t, O \Psi_t \rangle = \langle \Psi_{\text{gs}}, O \Psi_{\text{gs}} \rangle \quad (3.1)$$

We will also need the time-independent bound on the total number of photons.

Lemma 3.2 (Photon bound [DRK12]). *There is a $\kappa > 0$ such that*

$$\sup_t \langle \Psi_t, e^{\kappa N} \Psi_t \rangle \leq C < \infty \quad (3.2)$$

with C depending on $\Psi_0 \in \mathcal{D}_\alpha$ and κ .

The following result describes the localization of photons in the ground state.

Lemma 3.3 (Localization of ground state photons). *Let θ be a smooth indicator such that $\text{Supp}\theta \subset \{0 < |x|\}$. Then*

$$\langle \Psi_{\text{gs}}, d\Gamma(\theta_t) \Psi_{\text{gs}} \rangle = \mathcal{O}(\langle t \rangle^{-\alpha}) \quad (3.3)$$

The next lemma bounds the number of photons after time t in spatial regions around the atom.

Lemma 3.4 (Propagation bound [DRK14]). *Let θ be a smooth indicator such that $\text{Supp}\theta \subset \{|x| < 1\}$. Then for any time $t_c \geq |\lambda|^{-2}$ the limit $a(\theta, t_c) := \lim_{t \rightarrow \infty} \langle \Psi_t, d\Gamma(\theta_{t_c}) \Psi_t \rangle$ exists and moreover*

$$\langle \Psi_t, d\Gamma(\theta_{t_c}) \Psi_t \rangle = a(\theta, t_c) + \mathcal{O}(\langle t \rangle^{-\alpha}) \quad (3.4)$$

uniformly for $t_c \in [|\lambda|^{-2}, t]$.

In Section 4, we will identify

$$a(\theta, t) = \langle \Psi_{\text{gs}}, d\Gamma(\theta_t) \Psi_{\text{gs}} \rangle \quad (3.5)$$

Note that in the case where $0 \notin \text{Supp}\theta$, we can combine this propagation bound and (3.5) with Lemma 3.3 to conclude that

$$\langle \Psi_t, d\Gamma(\theta_t) \Psi_t \rangle = \mathcal{O}(\langle t \rangle^{-\alpha}) \quad (3.6)$$

Finally, we need a more detailed photon bound

Lemma 3.5 (Soft photon bound [DRK14]). *For any $\epsilon > 0$, we have*

$$\sup_t \langle \Psi_t, d\Gamma(\mathbb{1}_{|k| \leq \epsilon}) \Psi_t \rangle \leq C \epsilon^{\alpha/2} \quad (3.7)$$

with C depending on $\Psi_0 \in \mathcal{D}_\alpha$ and α , but not on ϵ .

4 Strong local relaxation

In Lemma 3.1, we saw that the system relaxes to the ground state in a weak sense. The following result expresses that, locally in space, it also relaxes in a strong sense. This will also allow us to identify the numbers $a(\theta, t)$ in Lemma 3.4.

Let $B(r) \subset \mathbb{R}^3$ be the ball with radius r in position space and $B^c(r) = \mathbb{R}^3 \setminus B(r)$. We will often split

$$\mathcal{H} = \mathcal{H}_{B(r)} \otimes \mathcal{H}_{B^c(r)} \quad (4.1)$$

where it is understood that the small system is incorporated in the inner sphere, i.e.

$$\mathcal{H}_{B(r)} := \mathcal{H}_S \otimes \Gamma(L^2(B(r))), \quad \mathcal{H}_{B^c(r)} := \Gamma(L^2(B^c(r))) \quad (4.2)$$

Let us then write in general $\text{Tr}_{B(r)}$ and $\text{Tr}_{B^c(r)}$ for the partial traces over $\mathcal{H}_{B(r)}$ and $\mathcal{H}_{B^c(r)}$, respectively.

Proposition 4.1 (Strong local relaxation). *Fix $r > 0$. Then*

$$\lim_{t \rightarrow \infty} \text{Tr}_{B^c(r)} |\Psi_t\rangle\langle\Psi_t| = \text{Tr}_{B^c(r)} |\Psi_{\text{gs}}\rangle\langle\Psi_{\text{gs}}| \quad (4.3)$$

Proof. Let $\Delta_{B(r)}$ be the Laplacian on the ball $B(r)$ with Neumann boundary conditions. Then, for any ψ in its form domain,

$$-\langle\psi, \Delta_{B(r)}\psi\rangle_{L^2(B(r))} = \int_{B(r)} dx |\nabla\psi(x)|^2 \quad (4.4)$$

and this form domain is in fact the Sobolev space $W^{2,1}(B(r))$ (completion of $C^\infty(B(r))$ w.r.t. the norm $\|\psi\| + \|\nabla\psi\|$). Hence, in the sense of quadratic forms on the form domain of $\Delta_{\mathbb{R}^d}$,

$$-\mathbb{1}_{B(r)}\Delta_{B(r)}\mathbb{1}_{B(r)} \leq -\Delta_{\mathbb{R}^d}. \quad (4.5)$$

For $\Psi \in \text{Dom}(H_F)$, we then get

$$\langle\Psi, H_F^2\Psi\rangle \geq \langle\Psi, d\Gamma(-\Delta_{\mathbb{R}^d})\Psi\rangle \geq \langle\Psi, d\Gamma(-\mathbb{1}_{B(r)}\Delta_{B(r)}\mathbb{1}_{B(r)})\Psi\rangle \quad (4.6)$$

Using the relative boundedness of H_I w.r.t to H_F , the boundedness of H_S and the fact that H^2 is conserved, we obtain

$$\sup_t \langle\Psi_t, H_F^2\Psi_t\rangle \leq C\|\Psi_0\|^2 + \|H\Psi_0\|^2. \quad (4.7)$$

Let $N_{B(r)} = d\Gamma(1_{B(r)})$, then by the photon number bound we have

$$\sup_t \langle\Psi_t, N_{B(r)}^2\Psi_t\rangle \leq C. \quad (4.8)$$

We can restate these two a priori bounds by saying that the orbit of the reduced density matrix

$$\rho_{t,r} := \text{Tr}_{B^c(r)} |\Psi_t\rangle\langle\Psi_t| \quad (4.9)$$

remains in

$$\mathcal{K} = \mathcal{K}_{C_N, C_\Delta} := \{\rho \in \mathcal{B}_1(\mathcal{H}_{B(r)}) \mid \text{Tr}[N_B^2 \rho] < C_N, \text{Tr}[\text{d}\Gamma(-\Delta_{B(r)})\rho] < C_\Delta\} \quad (4.10)$$

for some fixed C_N, C_Δ (that depend on the initial condition). By the compact Sobolev embedding, we easily find that the set \mathcal{K} is compact in the trace norm topology, and hence all sequences $\rho_{t_j, r}$ with $t_j \rightarrow \infty, j \rightarrow \infty$ have convergent subsequences. To argue that they necessarily converge to $\rho_{\infty, r} := \text{Tr}_{B^c(r)} |\Psi_{\text{gs}}\rangle\langle\Psi_{\text{gs}}|$, we need the Lemma 4.2 below. We now use this Lemma with ρ , a limit point of $\rho_{t, 2r}$, and with $\rho' = \text{Tr}_{B^c(2r)} |\Psi_{\text{gs}}\rangle\langle\Psi_{\text{gs}}|$. The hypothesis $\rho_r \neq \rho'_r$ leads to a contradiction because of the weak relaxation (Lemma 3.1). \square

Lemma 4.2. *Let $\rho, \rho' \in \mathcal{B}_1(\mathcal{H}_{B(2r)})$ be positive operators satisfying $\text{Tr} \rho N^2, \text{Tr} \rho' N^2 < \infty$, and denote by ρ_r, ρ'_r their respective restrictions (partial traces) to $\mathcal{B}_1(\mathcal{H}_{B(r)})$. If $\rho_r \neq \rho'_r$, then there is a $W \in \mathcal{W}_\alpha \cap \mathcal{B}(\mathcal{H}_{B(2r)})$ such that*

$$\text{Tr} \rho W \neq \text{Tr} \rho' W \quad (4.11)$$

Proof. Given a dense set $\mathfrak{g} \subset L^2(B(r))$, the C^* algebra generated by the elements $S \in \mathcal{B}(\mathcal{H}_{\mathfrak{S}})$ and $\mathcal{W}(\psi), \psi \in \mathfrak{g}$ separates points in $\mathcal{B}_1(\mathcal{H}_{B(r)})$. This does not immediately prove the lemma because $\mathfrak{h}_\alpha \cap L^2(B(r))$ need not be dense in $L^2(B(r))$. However, let us consider the dense set $1_{B(r)} \mathfrak{h}_\alpha$ and write $\psi = \psi_{B(r)} + \psi_{B^c(r)}$ with $\psi_{B(r)} = 1_{B(r)} \psi_{B(r)}$. Since (see, for example, [GD99])

$$\|(\mathcal{W}(\psi) - \mathbb{1})(1 + N)^{-1}\| \leq C \|\psi\|_{\mathfrak{h}} \quad (4.12)$$

and $\mathcal{W}(\psi) = \mathcal{W}(\psi_{B(r)})\mathcal{W}(\psi_{B^c(r)})$, we get

$$|\text{Tr} \rho \mathcal{W}(\psi) - \text{Tr} \rho \mathcal{W}(\psi_{B(r)})| \leq C \|\psi_{B^c(r)}\|^2 \|\mathcal{W}(\psi_{B(r)})\| \|\rho N\|_1 \quad (4.13)$$

and analogously for ρ' . This proves the claim because, given an element $\varphi \in 1_{B(r)} \mathfrak{h}_\alpha$ and $\epsilon > 0$, we can find $\psi \in \mathfrak{h}_\alpha$ such that $\varphi = \psi_{B(r)}$ and $\|\psi_{B^c(r)}\| \leq \epsilon$. \square

By combining the strong relaxation with the photon bound, it is clear that we also get relaxation of some unbounded observables on $\mathcal{H}_{B(r)}, r < \infty$. Indeed, let O be an operator on $\mathcal{H}_{B(r)}$ such that, say $\|O \mathbb{1}_{N=n}\|$ grows not faster than polynomially in n , then the second term in

$$\langle \Psi_t, O \Psi_t \rangle = \langle \Psi_t, O \mathbb{1}_{N \leq n} \Psi_t \rangle + \langle \Psi_t, O \mathbb{1}_{N > n} \Psi_t \rangle \quad (4.14)$$

can be bounded uniformly in time t by a number that vanishes as $n \rightarrow \infty$, and the first term converges as $t \rightarrow \infty$. It follows that

$$\lim_{t \rightarrow \infty} \langle \Psi_t, O \Psi_t \rangle = \langle \Psi_{\text{gs}}, O \Psi_{\text{gs}} \rangle \quad (4.15)$$

In particular, this allows us to make the identification (3.5) thus supplementing Lemma 3.4. To end this section, we give the

Proof of Lemma 2.4

If Lemma 2.4 were false, then for any $\kappa > 0$, $\limsup_{n \rightarrow \infty} a(n)e^{\kappa n} = \infty$, with $a(n) := \|\mathbb{1}_{N=n} \Psi_{\text{gs}}\|$. For any number m , we could then find $n \in \mathbb{N}$ and $r < \infty$ such that

$$\|\mathbb{1}_{N=n} \Gamma(1_{B(r)}(x)) \Psi_{\text{gs}}\| \geq m e^{-\kappa n} \quad (4.16)$$

Let $\kappa > 0$ be as in Lemma 3.2 and choose m such that

$$\sup_t \|\mathbb{1}_{N=n} \Psi_t\| \leq (m/2) e^{-\kappa n} \quad (4.17)$$

for some arbitrarily chosen Ψ_0 . Since $O = \mathbb{1}_{N=n} \Gamma(1_{B(r)}(x))$ is a bounded operator on $\mathcal{H}_{B(r)}$, Proposition 4.1 implies that $\langle \Psi_t, O \Psi_t \rangle \rightarrow \langle \Psi_{\text{gs}}, O \Psi_{\text{gs}} \rangle$, which is clearly incompatible with (4.16, 4.17).

5 Proof of the main result

5.1 Scattering identification and inverse

We introduce some standard tools used in scattering theory. In their present form, these tools were introduced in [GD99] and we follow very closely, if not literally, this source. First, we recall the natural isomorphism of Hilbert spaces

$$\Gamma(\mathfrak{h} \oplus \mathfrak{h}) \sim \Gamma(\mathfrak{h}) \otimes \Gamma(\mathfrak{h}) \quad (5.1)$$

Consider j_0, j_∞ bounded operators on \mathfrak{h} satisfying

$$j_0 + j_\infty = 1, \quad j_0^* j_0 + j_\infty^* j_\infty \leq \mathbb{1} \quad (5.2)$$

and define $j : \mathfrak{h} \rightarrow \mathfrak{h} \oplus \mathfrak{h} : \phi \mapsto (j_0 \phi, j_\infty \phi)$, satisfying $\|j\| \leq 1$ because of (5.2). We construct the operators $\check{\Gamma}(j) : \Gamma(\mathfrak{h}) \rightarrow \Gamma(\mathfrak{h}) \otimes \Gamma(\mathfrak{h})$ by

$$\check{\Gamma}(j) a^*(f_1) a^*(f_2) \dots a^*(f_m) \Omega \quad (5.3)$$

$$:= (a^*(j_0 f_1) \otimes \mathbb{1} + \mathbb{1} \otimes a^*(j_\infty f_1)) \dots (a^*(j_0 f_m) \otimes \mathbb{1} + \mathbb{1} \otimes a^*(j_\infty f_m)) (\Omega \otimes \Omega) \quad (5.4)$$

where Ω are Fock vacua. Note that $\|\check{\Gamma}(j)\| \leq 1$. Hence, the picture that goes with the above definition of $\check{\Gamma}(j)$ is that each photon is split into a part close to the origin and one part far from the origin. The former is put in the first copy of Fock space, the latter into the second.

Let us now pick a particular $j = j(t) : \mathfrak{h} \rightarrow \mathfrak{h} \oplus \mathfrak{h}$ defined by $j_0(t) = \theta(\cdot/t)$ and $j_\infty(t) = 1 - j_0(t)$ with θ a smooth indicator (see Section 3) such that $\text{Supp}\theta \subset \{|x| \leq v_2\}$ and $\theta = 1$ on $\{|x| \leq v_1\}$, with $v_1 < v_2 < 1$. It is clear that $j(t)$ satisfies the conditions (5.2). With this choice of $j(t)$, we construct an 'extended inverse scattering identification':

$$J_{\text{ex},t} := \mathbb{1} \otimes \check{\Gamma}(j(t)) : \mathcal{H} \rightarrow \mathcal{H}_{\text{as}} \quad (5.5)$$

where \mathcal{H}_{as} is the asymptotic space defined by

$$\mathcal{H}_{\text{as}} = \mathcal{H} \otimes \mathcal{H}_{\text{F}} \sim \mathcal{H}_{\text{S}} \otimes \mathcal{H}_{\text{F}} \otimes \mathcal{H}_{\text{F}} \quad (5.6)$$

The operator $J_{\text{ex},t}$ is a one-sided inverse to the 'extended scattering identification' $I_{\text{ex}} : \mathcal{H}_{\text{as}} \rightarrow \mathcal{H}$, see (5.8). To construct I_{ex} , define it first as a closable operator $\Gamma(\mathfrak{h} \oplus \mathfrak{h}) \sim \mathcal{H}_{\text{F}} \otimes \mathcal{H}_{\text{F}} \rightarrow \mathcal{H}_{\text{F}}$ by $\Gamma(\iota)$ with $\iota : \mathfrak{h} \oplus \mathfrak{h} \rightarrow \mathfrak{h} : (\phi, \phi') \mapsto \phi + \phi'$, or, in a more intuitive representation, by

$$I_{\text{ex}}(a^*(f_1) \dots a^*(f_n) \Omega \otimes a^*(f'_1) \dots a^*(f'_{n'}) \Omega) = a^*(f_1) \dots a^*(f_n) a^*(f'_1) \dots a^*(f'_{n'}) \Omega \quad (5.7)$$

and then extend by tensoring with $\mathbb{1}$ on \mathcal{H}_{S} . Note that $\iota j(t) = \mathbb{1}$ and hence indeed

$$I_{\text{ex}} J_{\text{ex},t} = \mathbb{1}. \quad (5.8)$$

The operator I_{ex} is unbounded, but by using $\|a(f)\mathbb{1}_{N=n}\| = \sqrt{n}\|f\|_{\mathfrak{h}}$, one can compute

$$\|I_{\text{ex}}(\mathbb{1}_{N < n'} \otimes \mathbb{1}_{N < n})\|^2 = \frac{(n+n')!}{n!n'!} \quad (5.9)$$

I_{ex} is related to I by

$$I_{\text{ex}} \Psi_{\text{gs}} \otimes \Psi_{\text{F}} = I \Psi_{\text{F}} \quad (5.10)$$

For our analysis, it is important that $I\mathbb{1}_{N \leq n}$ is bounded for any n , as follows by combining (5.9) and Lemma 2.4. In particular, this implies that the limit $W_t \Psi$ in (2.14) exists for *any* $\Psi \in \mathcal{D}_{\text{fin}}$. Finally, we also define

$$J_t := \langle \Psi_{\text{gs}} | \otimes \mathbb{1} J_{\text{ex},t} : \mathcal{H} \rightarrow \mathcal{H}_{\text{F}} \quad (5.11)$$

and we will argue that J_t is an approximative inverse to I .

5.2 Proof of main result

We define the asymptotic Hamiltonian H_{as} on \mathcal{H}_{as} as

$$H_{\text{as}} = H \otimes \mathbb{1} + \mathbb{1} \otimes H_{\text{F}} \quad (5.12)$$

The following two propositions are our main technical results. To state these results, and also in the remainder of the paper, we assume that Assumptions 2.1 and 2.2 hold and that $|\lambda|$ is sufficiently small. This will not be repeated any more.

Proposition 5.1 (Range of inverse wave operator). *For $\Psi \in \mathcal{D}_\alpha$,*

$$\lim_{n \rightarrow \infty} \limsup_{t \rightarrow +\infty} \left\| (I \mathbb{1}_{[N \leq n]} J_t - \mathbb{1}) e^{-itH} \Psi \right\| = 0. \quad (5.13)$$

Proposition 5.2 (Existence of inverse wave operator). *The limit*

$$Z_{\text{ex}} \Psi := \lim_{t \rightarrow +\infty} e^{itH_{\text{as}}} J_{\text{ex},t} e^{-itH} \Psi \quad (5.14)$$

exists for $\Psi \in \mathcal{D}_\alpha$. Note that Z_{ex} extends to a contraction $\mathcal{H} \rightarrow \mathcal{H}_{\text{as}}$ because J_{ex} is a contraction.

The remaining sections of the paper are devoted to the proof of Propositions 5.1 and 5.2. We now show how they lead to our main result.

Proof of Theorem 2.6. Define the contraction $Z : \mathcal{H} \rightarrow \mathcal{H}_{\text{F}}$ by

$$Z\Psi := \langle \Psi_{\text{gs}} | \otimes \mathbb{1} (Z_{\text{ex}} \Psi) = \lim_{t \rightarrow +\infty} e^{it(E_{\text{gs}} + H_{\text{F}})} J_t e^{-itH} \Psi. \quad (5.15)$$

Using this and the fact that $\lim_{t \rightarrow \infty} W_t \Psi = W_+ \Psi$ holds for any $\Psi \in \mathcal{D}_{\text{fin}}$ (Theorem 2.5 combined with the remark following (5.9)), we get

$$\lim_{t \rightarrow +\infty} e^{itH} I e^{-it(E_{\text{gs}} + H_{\text{F}})} \mathbb{1}_{[N \leq n]} e^{it(E_{\text{gs}} + H_{\text{F}})} J_t e^{-itH} \Psi = W_+ \mathbb{1}_{[N \leq n]} Z\Psi \quad (5.16)$$

Since W_+ is an isometry, we can take the limit $n \rightarrow \infty$ on the RHS and obtain $W_+ Z\Psi$. By Proposition 5.1, the $n \rightarrow \infty$ limit of the LHS equals Ψ . Since \mathcal{D}_α is dense, we have proven that $W_+ Z$ extends to the identity on \mathcal{H} , hence in particular $\text{Ran} W_+ = \mathcal{H}$. \square

6 Range of the wave operator

In this section, we prove Proposition 5.1. Fix a smooth indicator $\theta^{(1)}$ such as described in Section 3. Let $0 < v_1 < v_2 < 1$ be such that $\theta^{(1)}(x) = 1$ for $|x| \leq v_1$ and 0 for $|x| \geq v_2$. The indicator $\theta^{(1)}$ is used to construct the identification

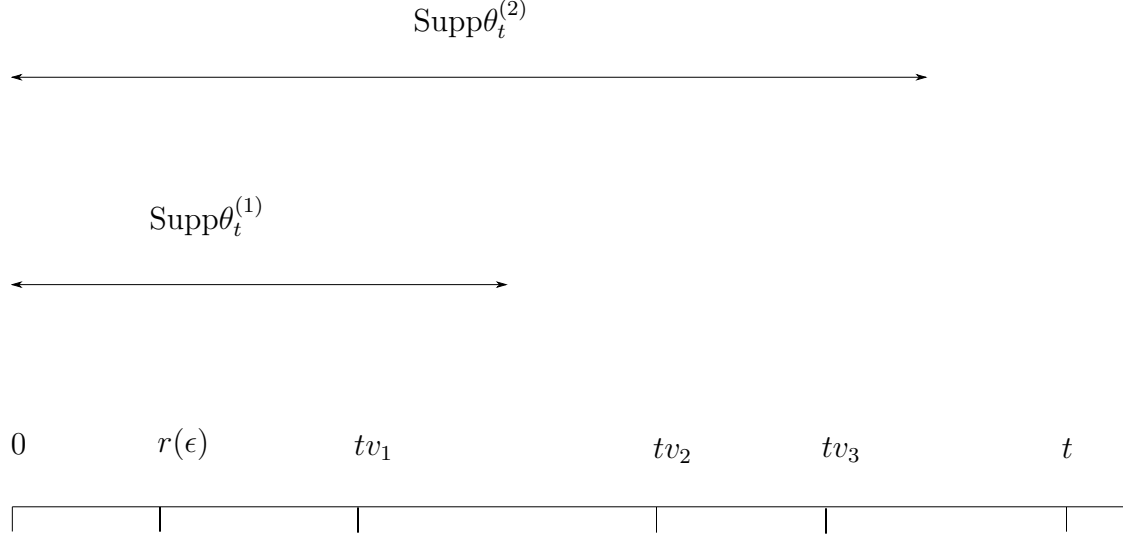


Figure 1: Important distances from the origin to the position of the outgoing waves, at $r \sim t$.

operator J_t (see Section 5). Moreover, we need another smooth indicator $\theta^{(2)}$ such that $\theta^{(2)}(x) = 1$ for $|x| \leq v_2$ but still $\text{Supp}\theta^{(2)} \subset B(v_3)$, for some $v_3 < 1$. The relevant distances are summarized in Figure 1. Note that for a measurable $K \subset \mathbb{R}^d \setminus \{0\}$, the space \mathcal{H}_K is Fock space in a natural way, namely

$$\mathcal{H}_K = \Gamma(L^2(K)) \quad (6.1)$$

and we denote the corresponding vacuum by Ω_K . We also write $N_K = d\Gamma(1_K)$ for the photon number in the region K .

We define

$$\Psi_{\text{gs}, B(r)} := \langle \Omega_{B^c(r)} | \Psi_{\text{gs}} \rangle \in \mathcal{H}_{B(r)}. \quad (6.2)$$

Then $\Psi_{\text{gs}, B(r)} \otimes \Omega_{B^c(r)}$ is an approximation to the ground state, more precisely

$$\|\Psi_{\text{gs}, B(r)} \otimes \Omega_{B^c(r)} - \Psi_{\text{gs}}\| \rightarrow 0, \quad r \rightarrow \infty \quad (6.3)$$

Quantities depending on $\epsilon > 0$

We fix a small $\epsilon > 0$ and we now choose $r(\epsilon), t(\epsilon)$ as follows: We first choose $r(\epsilon)$ such that for $r > r(\epsilon)$,

$$\left| \|\langle \Psi_{\text{gs}, B(r)} | \Psi_{\text{gs}} \rangle\|^2 - 1 \right| \leq \epsilon, \quad \langle \Psi_{\text{gs}}, N_{B^c(r)} \Psi_{\text{gs}} \rangle \leq \epsilon, \quad (6.4)$$

then we choose $t(\epsilon)$ such that for $t > t(\epsilon)$ and $r = r(\epsilon)$:

1.

$$\left| \|\langle \Psi_{\text{gs}, B(r)} | \Psi_t \|^2 - 1 \right| \leq \epsilon, \quad \left| \langle \Psi_{\text{gs}}, N_{B(r)} \Psi_{\text{gs}} \rangle - \langle \Psi_t, N_{B(r)} \Psi_t \rangle \right| \leq \epsilon. \quad (6.5)$$

This is possible by Proposition 4.1 and the remarks following it.

2. The error term $\mathcal{O}(t^{-\alpha})$ in Lemma 3.4 with $\theta = \theta^{(1)}$ and $\theta = \theta^{(2)}$ is not larger than ϵ .

For $r = r(\epsilon)$ and $t \geq t(\epsilon)$, as introduced above, we define the time-dependent annular region

$$M_t = B(tv_2) \setminus B(r). \quad (6.6)$$

Our proof of Proposition 5.1 exploits that there are no photons in M (up to $\mathcal{O}(\epsilon)$ -terms). Indeed we have:

Lemma 6.1. *For $r = r(\epsilon)$ and $t \geq t(\epsilon)$:*

$$\|\langle \Psi_{\text{gs}, B(r)} \otimes \Omega_{M_t} | \Psi_t \rangle\| = 1 + \mathcal{O}(\epsilon) \quad (6.7)$$

Proof. From $1_{B(r)} + 1_M \leq \theta_t^{(2)}$ we get

$$\langle \Psi_t, N_{M_t} \Psi_t \rangle + \langle \Psi_t, N_{B(r)} \Psi_t \rangle \leq \langle \Psi_t, d\Gamma(\theta_t^{(2)}) \Psi_t \rangle. \quad (6.8)$$

In $\langle \Psi_t, N_{B(r)} \Psi_t \rangle$ and $\langle \Psi_t, d\Gamma(\theta_t^{(2)}) \Psi_t \rangle$ we now replace the vector Ψ_t by Ψ_{gs} using our choices for r and $t(\epsilon)$ expressed in Points 1. and 2. above. Then, using $d\Gamma(\theta_t^{(2)}) \leq N_{B(r)} + N_{B^c(r)}$ as well as (6.4) we arrive at

$$\langle \Psi_t, N_{M_t} \Psi_t \rangle \leq \langle \Psi_{\text{gs}}, N_{B^c(r)} \Psi_{\text{gs}} \rangle + 2\epsilon \leq 3\epsilon, \quad (6.9)$$

Let us now define the projectors on \mathcal{H}

$$P_1 = \frac{|\Psi_{\text{gs}, B(r)}\rangle\langle\Psi_{\text{gs}, B(r)}|}{\|\Psi_{\text{gs}, B(r)}\|^2} \otimes \mathbb{1}, \quad P_2 = \mathbb{1} \otimes |\Omega_{M_t}\rangle\langle\Omega_{M_t}| \otimes \mathbb{1} \quad (6.10)$$

and write $P^\perp = \mathbb{1} - P$ for projectors P , then we get (for $j = 1$ from (6.4) and for $j = 2$ from the above)

$$\langle \Psi_t, P_j^\perp \Psi_t \rangle \leq C\epsilon \quad (6.11)$$

and therefore, since $(P_1 P_2)^\perp \leq P_1^\perp + P_2^\perp$,

$$\langle \Psi_t, (P_1 P_2)^\perp \Psi_t \rangle \leq C\epsilon \quad (6.12)$$

which yields the lemma. \square

Lemma 6.2. Consider Hilbert spaces $\mathcal{H}_a, \mathcal{H}_b$. Let $\Psi \in \mathcal{H}_a \otimes \mathcal{H}_b$, $\|\Psi\| = 1$ and let $\Psi_a \in \mathcal{H}_a, \Psi_b \in \mathcal{H}_b$ be such that

$$\Psi_b = \langle \Psi_a | \Psi \rangle, \quad |1 - \|\Psi_a\|^2|, |1 - \|\Psi_b\|^2| \leq \nu. \quad (6.13)$$

Then $\|\Psi - \Psi_a \otimes \Psi_b\|^2 \leq C\nu$.

Proof. By the definition of Ψ_b , $\langle \Psi, \Psi_a \otimes \Psi_b \rangle = \|\Psi_b\|^2$ and hence

$$\langle \Psi - \Psi_a \otimes \Psi_b, \Psi - \Psi_a \otimes \Psi_b \rangle = 1 + \|\Psi_a\|^2 \|\Psi_b\|^2 - 2\|\Psi_b\|^2 \quad (6.14)$$

□

Proof of Proposition 5.1

We will apply Lemma 6.2 with

$$\Psi = \Psi_t, \quad \Psi_a = \Psi_{\text{gs}, B(r)} \otimes \Omega_{M_t}, \quad \Psi_b = \langle \Psi_a | \Psi_t =: \Psi_{B^c(tv_2), t}.$$

To recall our conventions, note that $B(tv_2) = M_t \cup B(r)$ and

$$\Psi_{\text{gs}, B(r)} \otimes \Omega_{M_t} \in \mathcal{H}_{B(r)} \otimes \mathcal{H}_{M_t} = \mathcal{H}_{B(tv_2)}, \quad \Psi_{B^c(tv_2), t} \in \mathcal{H}_{B^c(tv_2)}.$$

By Lemma 6.1, we can take $\nu = C\epsilon$ and then Lemma 6.2 yields

$$\|\Psi_t - \Psi_{\text{gs}, B(r)} \otimes \Omega_{M_t} \otimes \Psi_{B^c(tv_2), t}\| \leq C\sqrt{\epsilon} \quad (6.15)$$

By the uniform in time photon bound, we have

$$\sup_t \langle \Psi_{B^c(tv_2), t}, N \Psi_{B^c(tv_2), t} \rangle \leq C. \quad (6.16)$$

Since the region where $\theta_t^{(1)}$ is nonconstant is in $B(tv_2) \setminus B(tv_1) \subset M_t$, we have

$$J_{\text{ex}, t} \Psi_{\text{gs}, B(r)} \otimes \Omega_{M_t} \otimes \Psi_{B^c(tv_2), t} = (\Psi_{\text{gs}, B(r)} \otimes \Omega_{B^c(r)}) \otimes (\Omega_{B(tv_2)} \otimes \Psi_{B^c(tv_2), t}). \quad (6.17)$$

Therefore, by (6.15) and the choice of r , i.e. (6.4),

$$\sup_{t \geq t(\epsilon)} \|J_t \Psi_t - \Omega_{B(tv_2)} \otimes \Psi_{B^c(tv_2), t}\| \leq C\sqrt{\epsilon} \quad (6.18)$$

(keep in mind that we have constrained $t \geq t(\epsilon)$ throughout this section). Next, let us consider the map

$$R_{n, r'} : \mathbb{1}_{N \leq n} \mathcal{H}_{B^c(r')} \rightarrow \mathcal{H} : \Psi \mapsto I(\Omega_{B(r')} \otimes \Psi) - \Psi_{\text{gs}, B(r')} \otimes \Psi \quad (6.19)$$

Then, for any n , we have $\lim_{r' \rightarrow \infty} \|R_{n,r'}\| = 0$ by Lemma 2.4 and the bound (5.9). Since $\sup_t \|\Psi_{B^c(tv_2),t}\| \leq C$, this yields

$$\lim_{t \rightarrow +\infty} \|I(\Omega_{B(tv_2)} \otimes \mathbb{1}_{N \leq n} \Psi_{B^c(tv_2),t}) - \Psi_{\text{gs},B(tv_2)} \otimes \mathbb{1}_{N \leq n} \Psi_{B^c(tv_2),t}\| = 0 \quad (6.20)$$

Combining this with (6.18) and the fact that $\|I\mathbb{1}_{N \leq n}\| < \infty$, we get

$$\lim_{t \rightarrow +\infty} \|I\mathbb{1}_{N \leq n} J_t \Psi_t - \Psi_{\text{gs},B(tv_2)} \otimes \mathbb{1}_{N \leq n} \Psi_{B^c(tv_2),t}\| \leq C\sqrt{\epsilon} \quad (6.21)$$

Next, we argue that

$$\lim_{n \rightarrow +\infty} \limsup_{t \rightarrow +\infty} \|\Psi_t - \Psi_{\text{gs},B(tv_2)} \otimes \mathbb{1}_{N \leq n} \Psi_{B^c(tv_2),t}\| \leq C\sqrt{\epsilon} \quad (6.22)$$

Indeed, by our choice of $r(\epsilon)$, i.e. (6.4), we have

$$\sup_{t \geq t(\epsilon)} \|\Psi_{\text{gs},B(tv_2)} - \Psi_{\text{gs},B(r)} \otimes \Omega_{M_t}\| \leq C\epsilon \quad (6.23)$$

and by the photon bound (6.16) we have $\lim_{n \rightarrow +\infty} \sup_t \|\mathbb{1}_{N > n} \Psi_{B^c(tv_2),t}\| = 0$. Therefore, (6.22) follows from (6.15).

Since ϵ can be chosen arbitrarily small, the limits (6.21) and (6.22) equal 0. Together, they imply Proposition 5.1 by the triangle inequality.

7 Existence of wave operator

In this section, we prove Proposition 5.2. We calculate (the formal manipulations are justified since $\|N\Psi_t\| < \infty$)

$$i \frac{d}{dt} e^{itH_{\text{as}}} J_t e^{-itH} \Psi = e^{itH_{\text{as}}} (\mathbb{1}_S \otimes B_t) \Psi_t + e^{itH_{\text{as}}} \left((1 \otimes H_I) \check{\Gamma}(j) - \check{\Gamma}(j) H_I \right) \Psi_t \quad (7.1)$$

where

$$B_t = (d\Gamma(\omega) \otimes 1 + 1 \otimes d\Gamma(\omega)) \check{\Gamma}(j) - \check{\Gamma}(j) d\Gamma(\omega) - i \frac{d}{dt} \check{\Gamma}(j) \quad (7.2)$$

To estimate the second term in (7.1), we remark that it is bounded by

$$C \|a^\#(j_\infty \phi) \Psi_t\| \leq C \|j_\infty \phi\| \|N^{1/2} \Psi_t\| \leq C \|j_\infty \phi\| \quad (7.3)$$

where we used the photon bound. By Assumption 2.1 and the remark following it, we can deduce that $\|j_\infty \phi\| \leq \langle t \rangle^{-(1+\gamma)}$ for some $\gamma > 0$, and therefore this term is integrable. We focus on the first term in (7.1). From Lemma 2.16. in [GD99], we have the bound

$$\|B_t \Psi_t\| \leq 2 \|N^{1/2} \Psi_t\| \langle \Psi_t, d\Gamma(w_0^* w_0) \Psi_t \rangle^{1/2} \quad (7.4)$$

where

$$w_0 = i[\omega, j_0] - \frac{d}{dt} j_0. \quad (7.5)$$

Because of the photon bound in Lemma 3.2, Proposition 5.2 will be proven if we show that

$$\langle \Psi_t, d\Gamma(w_0^* w_0) \Psi_t \rangle^{1/2} \leq C \langle t \rangle^{-(1+\gamma)} \quad (7.6)$$

for some $\gamma > 0$. In the following the Lemma 7.1, below, is used repeatedly.

7.1 Bound on $\langle \Psi_t, d\Gamma(w_0^* w_0) \Psi_t \rangle^{1/2}$

We use the results of Appendix C to write

$$w_0 = \frac{1}{t} \left(\frac{k}{|k|} - \frac{x}{t} \right) \cdot \nabla \theta(x/t) + \langle t \rangle^{-1-\beta} b_t + \langle t \rangle^{-1} b'_t \mathbb{1}_{\{|k| \leq t^{\beta-1}\}} \quad (7.7)$$

where the exponent β satisfies $0 < \beta < 1$ and b_t, b'_t are operators that are bounded uniformly in t . By using Cauchy-Schwartz and Lemma 7.1, it suffices to prove that

$$\langle \Psi_t, d\Gamma(w_{0,i}^* w_{0,i}) \Psi_t \rangle = \mathcal{O}(t^{-2-\gamma}), \quad \gamma > 0 \quad (7.8)$$

where $w_{0,i}, i = 1, 2, 3$ is any of the three terms on the RHS of (7.7). We discuss the three terms separately.

The first term

Let

$$a_t = \left(\frac{k}{|k|} - \frac{x}{t} \right) \cdot \nabla \theta(x/t) \quad (7.9)$$

Since $\nabla \theta$ has support in $B(1)$ and $\| \left(\frac{k}{|k|} - \frac{x}{t} \right) 1_{[|x/t| \leq 1]} \| \leq C$, we can bound

$$a_t^* a_t \leq C |\nabla \theta(x/t)|^2 \quad (7.10)$$

Therefore, Lemma 7.1 and linearity of $b \rightarrow d\Gamma(b)$ yield the bound

$$\langle \Psi_t, d\Gamma(a_t^* a_t) \Psi_t \rangle \leq C \langle \Psi_t, d\Gamma(\tilde{\theta}(x/t) \Psi_t) \rangle, \quad \tilde{\theta}(x) := \frac{|\nabla \theta(x)|^2}{\sup_{x \in B(1)} |\nabla \theta(x)|^2} \quad (7.11)$$

Since $\tilde{\theta}$ is a smooth indicator whose support does not include 0, we can apply the bound 3.6 with θ replaced by $\tilde{\theta}$ to get the bound $\mathcal{O}(t^{-\alpha})$ for (7.11). This settles the first term in (7.7).

The second term

By Lemma 7.1,

$$\langle \Psi_t, d\Gamma(b_t^* b_t) \Psi_t \rangle \leq \|b_t^* b_t\| \langle \Psi_t, N \Psi_t \rangle \leq C \quad (7.12)$$

where the last inequality follows by the uniform boundedness of b_t and the photon bound, Lemma 3.2. This suffices for the second term, since we have $\beta > 0$.

The third term

Again Lemma 7.1 gives

$$d\Gamma(1_{[|k| \leq t^{\beta-1}]} (b_t')^* b_t' 1_{[|k| \leq t^{\beta-1}]}) \leq \| (b_t')^* b_t' \| d\Gamma(1_{[|k| \leq t^{\beta-1}]}) \quad (7.13)$$

so we just have to bound

$$\langle \Psi_t, d\Gamma(1_{[|k| \leq t^{\beta-1}]} \Psi_t) \rangle \quad (7.14)$$

We invoke Lemma 3.5 to bound the last expression by $\mathcal{O}(t^{\alpha(\beta-1)/2})$. Since $0 < \beta < 1$, this is a negative power of t .

The following lemma has been used repeatedly. Its elementary proof can be found e.g. in [DG99].

Lemma 7.1. *Let a, b be bounded operators on \mathfrak{h} .*

1) If $a \leq b$, then

$$d\Gamma(a) \leq d\Gamma(b) \quad (7.15)$$

In particular $d\Gamma(a) \leq \|a\|N$ for any self-adjoint a .

2)

$$|\langle \Psi d\Gamma(ab) \Psi \rangle| \leq \langle \Psi d\Gamma(aa^*) \Psi \rangle^{1/2} \langle \Psi d\Gamma(b^*b) \Psi \rangle^{1/2} \quad (7.16)$$

A Commutator estimates

Let $f \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ and $f_t(x) := f(x/t)$. Denote by f_t the corresponding multiplication operator on $L^2(\mathbb{R}^d)$ and by ω be the operator on $L^2(\mathbb{R}^d)$ given by the Fourier multiplier $\omega(k) = |k|$. We want to study the commutator $[\omega, f_t]$:

Lemma A.1. *Let $\nabla\omega$ be the Fourier multiplier $\nabla\omega(k) = \hat{k} \equiv k/|k|$. Then*

$$i[\omega, f_t] = \nabla\omega \cdot \nabla f_t + O_t \quad (C-1)$$

where, for $\beta > 0$

$$\|O_t \mathbb{1}_{[t^{\beta-1} \leq |k|]}\| \leq Ct^{-1-\beta}, \quad \|O_t\| \leq Ct^{-1}. \quad (C-2)$$

Remark. If $\omega(k)$ were C^2 a standard result (e.g. Lemma 27 in [FGS02b]) gives

$$\|O_t\| \leq t^{-2} \|\partial^2 \omega\|_\infty \int dq |\hat{f}(q)| |q|^2 \quad (C-3)$$

In our case $|\partial^2 \omega(p)| \sim \frac{1}{|p|}$ and this result is not applicable.

Proof. Let $\phi \in L^2(\mathbb{R}^d)$. We have

$$(\widehat{[\omega, f_t]\phi})(k) = \int dp \hat{f}_t(p) (\omega(k) - \omega(k-p)) \hat{\phi}(k-p).$$

Write

$$\omega(k) - \omega(k-p) = |k| - |k-p| = p \cdot \hat{k} + r(k, p).$$

We have, for $|p| \leq |k|/2$

$$|r(k, p)| = ||k| - |k-p| - p \cdot \hat{k}| \leq Cp^2/|k|$$

and for $|p| \geq |k|/2$, $|r(k, p)| \leq C|p|$. Hence altogether

$$|r(k, p)| \leq C \frac{p^2}{|p| + |k|}.$$

We obtained

$$i[\omega, f_t] = \hat{k} \cdot \nabla f_t + R$$

with

$$|\widehat{(R\phi)}(k)| \leq C \int dp |\hat{f}_t(p)| \frac{p^2}{|p| + |k|} |\hat{\phi}(k - p)|.$$

Hence

$$\|R\phi\|^2 \leq \int dp_1 dp_2 dk \prod_{i=1}^2 \hat{f}_t(p_i) \frac{p_i^2}{|p_i| + |k|} |\hat{\phi}(k - p_i)|.$$

Note that $\hat{f}_t(p) = t^d \hat{f}(tp)$ with \hat{f} rapidly decreasing. Thus, for $n \in \mathbb{N}$

$$\int dp |\hat{f}_t(p)| p^n \leq C(n) t^{-n}. \quad (\text{C-4})$$

First, we use $\frac{p_i^2}{|p_i| + |k|} \leq |p_i|$ to bound

$$\|R\phi\|^2 \leq \int dp_1 dp_2 \prod_{i=1}^2 \hat{p}_i \hat{f}_t(p_i) \int dk |\hat{\phi}(k - p_1)| |\hat{\phi}(k - p_2)| \leq C t^{-2} \|\phi\|^2 \quad (\text{C-5})$$

where we used (C-4) with $n = 1$ and Cauchy-Schwartz. This yields the second bound of (C-2). For ϕ such that $\text{Supp } \hat{\phi} \subset \{|k| \geq 1/t^{1-\beta}\}$, we use $\frac{p_i^2}{|p_i| + |k|} \leq |p_i|^2/|k|$, (C-4) with $n = 2$ and $|k|^{-1} \leq t$, to get

$$\|R\phi\|^2 \leq C t^{-2-2\beta} \|\phi\|^2, \quad (\text{C-6})$$

which gives the first bound of (C-2). □

We also need

Lemma A.2. *Let*

$$O_t := [\hat{k}, f_t(x)], \quad (\text{C-7})$$

Then for $\beta > 0$

$$\|O_t \mathbb{1}_{[t^{\beta-1} \leq |k|]}\| \leq C t^{-\beta}, \quad \|O_t\| \leq C. \quad (\text{C-8})$$

Proof. We have

$$\widehat{(O_t \phi)}(k) = \int dp \hat{f}_t(p) q(k, p) \hat{\phi}(k - p).$$

with

$$|q(k, p)| = \left| \frac{k}{|k|} - \frac{k - p}{|k - p|} \right| \leq \frac{|p|}{|p| + |k|}.$$

Now proceed as in the previous lemma. □

B Localization of ground state photons

Here we prove Lemma 3.3. By the choice of θ_t , we have for some $C > 0$ and all $x \in \mathbb{R}^3$,

$$0 \leq \theta_t(x) \leq C|t|^{-\alpha}|x|^\alpha. \quad (\text{E-1})$$

It follows that

$$\langle \Psi_{\text{gs}}, d\Gamma(\theta_t)\Psi_{\text{gs}} \rangle \leq C|t|^{-\alpha} \int |x|^\alpha \|\Psi_{\text{gs}}(x)\|^2 dx \quad (\text{E-2})$$

where $\Psi_{\text{gs}}(x)$ denotes the (inverse) Fourier transform of $a_k\Psi_{\text{gs}}$. Hence it remains to prove that (E-2) is finite. By a well-know pull-through trick [BFS98, Gér00],

$$a(k)\Psi_{\text{gs}} = \lambda\hat{\phi}(k)(H - E + |k|)^{-1}D\Psi_{\text{gs}}. \quad (\text{E-3})$$

This implies, by Assumption 2.1 on ϕ , that for some $\beta > \alpha$,

$$\|\partial_k^m a(k)\Psi_{\text{gs}}\| \leq C|k|^{(\beta-3)/2-|m|}, \quad |m| \leq 3.$$

The finiteness of (E-2) now follows from a straightforward extension of Lemma A.1 in [DRK14] to functions taking values in a Hilbert space. This finishes the proof of Lemma 3.3.

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